Factoring Notes

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1 Standard Rules

Generally, when solving a polynomial for its respective zeros (roots, x-ints) you will set the equation equal to zero and solve for the unknown variable. We have summarized the rules.

1. Set the equation to zero.
2. Bring out any common factor.
3. When possible, break down into terms.
4. Solve for roots by setting each set of terms to zero separately.

1.1 Common Factors

Bringing out a common factor involves finding a numerical value, even sometimes a variable, that is pulled out. This 'term' can later be multiplied back to get the same equation. A crude example is exhibited here with $a \neq 0$.

$$\begin{aligned} f(x) = ax^2 - bxa &= 0 \\ ax(x-b) &= 0 \\ ax = 0, \quad x - b &= 0 \\ x &= 0, b \end{aligned}$$

1.2 Difference of Two Squares

We may describe the difference of two squares. The difference of two squares is exactly what it sounds like. Note: $a \neq 0$.

$$\begin{aligned} f(x) = x^2 - a^2 &= 0 \\ x^2 - a^2 &= 0 \\ (x-a)(x+a) &= 0 \\ x - a = 0, \quad x + a &= 0 \\ x &= a, -a \end{aligned}$$

1.3 Factoring Trinomials

What is a trinomial? Trinomial is a three-term equation. Typically, of the second degree. The degree means the largest exponent of all the terms in the polynomial.

1.3.1 Second Degree Polynomials

Factoring a second-degree polynomial is an important skill to have when solving problems involving quadratic equations. Note: $B, C \neq 0$ and $B = (a + b)$, $C = ab$.

$$
\begin{aligned}
f(x) = x^2 + Bx + C &= 0 \\
x^2 + (a+b)x + ab &= 0 \\
(x+a)(a+b) &= 0 \\
x + a = 0, \quad x + b &= 0 \\
x &= -a, -b
\end{aligned}
$$

1.3.2 Factoring Perfect Squares

Perfect squares are different in the sense you ignore the middle term of the trinomial in order to find terms of the factors. You only need to look at the inner middle term when determining the sign of the perfect square. In essence, you look at the constant and the leading term for your factors. Let us demonstrate. Note: $a, b, c \neq 0$

$$
\begin{aligned}
f(x) = a^2x^2 + bx + c^2 &= 0 \\
(ax + c)(ax + c) &= 0 \\
(ax + c)^2 &= 0 \\
ax + c &= 0 \\
x &= -\frac{c}{a}
\end{aligned}
$$

1.4 Completing the Square

The quadratic equation is proven using the concept of completing the square. Typically, this is more useful when trying to simplify an equation into something we can understand before we factor. Simply, we have a polynomial $f(x) = ax^2 + bx + c$ that we may not be able to factor.

$$
\begin{aligned}
ax^2 + bx + c &= 0, \\
ax^2 + bx &= -c, \\
x^2 + \frac{b}{a}x &= -\frac{c}{a}, \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= -\frac{c}{a} + \frac{b^2}{4a^2}, \\
\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}, \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}, \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Here, you see that we have proven the quadratic equation using the concept of completing the square. Otherwise, we may rewrite the equation with a more generic definition of what complete the square means for the reader.

$$
\begin{aligned}
ax^2 + bx + c &= 0, \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0, \\
x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 + \frac{c}{a} &= (\frac{b}{2a})^2
\end{aligned}
$$

After that, of course you would manipulate the equation to the specific form you desire. Typically, this is useful for not only factoring, but finding vertex form of a quadratic or even a circle.

1.5 Quadratic Formula

Now, when do we use the venerable quadratic formula of time old? Any time you may not factor the given polynomial. This is typically when the polynomial is considered 'prime'.

Usually, a good start with a trinomial in standard form set to zero, $ax^2 + bx + c = 0$, and plug 'n' play from here. Note: $a, b, c \in \mathbb{R}$ and $a \neq 0$.

$$
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

It is important to note the discriminant of the quadratic equation set to zero, $ax^2 + bx + c = 0$, is $b^2 - 4ac$. This information helps us determine what solutions we will have below

- If $b^2 - 4ac < 0$ then $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$ then $ax^2 + bx + c = 0$ has exactly one solution.
- If $b^2 - 4ac > 0$ then $ax^2 + bx + c = 0$ has two real solutions.